The directed Oberwolfach problem with two tables

Alice Lacaze-Masmonteil, University of Ottawa
Joint work with Daniel Horsley, Monash University

December 11, 2023

Acknowledgements

- Daniel Horsley;
- Monash Department of Mathematics;
- Natural Sciences and Engineering Council of Canada.

NSERC
CRSNG

A simple example

The setting: Consider a conference with 12 participants. To facilitate networking, the organizing committee decides to host 11 banquets. The banquet hall has 2 tables that seat 4 and 8 participants.

The problem: The organizing committee needs a set of 11 seating arrangements (one for each banquet) such that each participant is seated to the right of every other participants exactly once.

Is this possible?

Construction of a seating arrangement

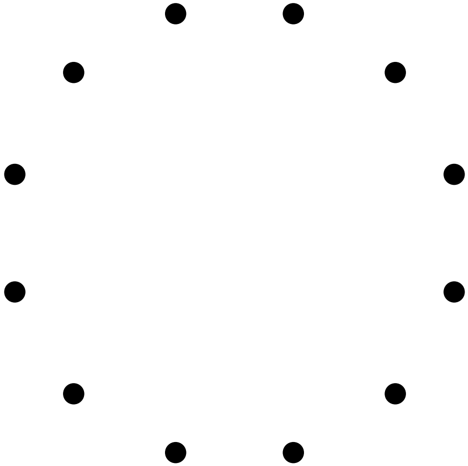

Figure: The 12 participants (one for each vertex).

Construction of a seating arrangement

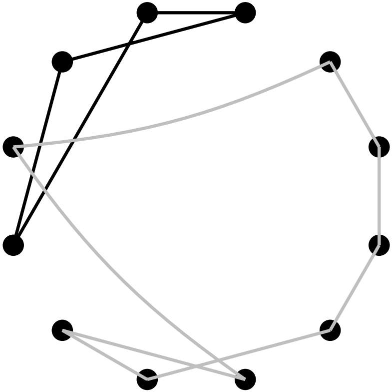

Figure: One seating arrangement with one table of length 4 and one table of length 8.

Construction of a seating arrangement

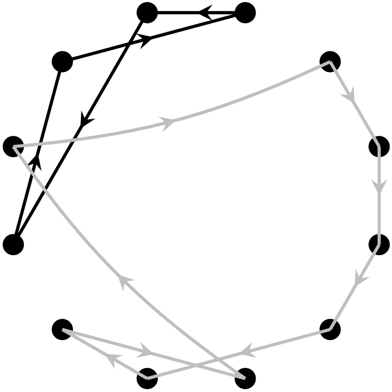

Figure: One seating arrangement with one table of length 4 and one table of length 8.

Construction of a seating arrangement

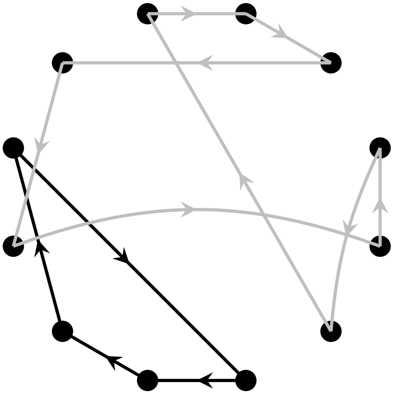

Figure: Another seating arrangement with one table of length 4 and one table of length 8.

The directed Oberwolfach problem

The setting: Consider a conference with n participants. To facilitate networking, the organizing committee decides to host $n-1$ banquets. The banquet hall has t round tables that sit $m_1, m_2, \ldots, m_t$ participants such that $m_1 + m_2 + \ldots + m_t = n$.

The problem: The organizing committee needs a set of $n-1$ seating arrangements (one for each banquet) such that each participant is seated **to the right** of every other participants exactly once.

Is this possible?

The complete symmetric digraph

Definition

Given a graph H, its **directed symmetric counterpart** *is the digraph obtained by replacing each edge of H with a pair of arcs (one for each direction).*

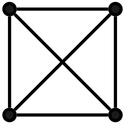

Figure: The complete graph K_4.

The complete symmetric digraph

Definition

Given a graph H, its **directed symmetric counterpart** *is the digraph obtained by replacing each edge of H with a pair of arcs (one for each direction).*

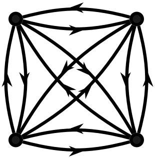

Figure: The complete symmetric digraph K_4^*.

The complete symmetric digraph K_n^* is the directed symmetric counterpart of K_n.

Definitions

Definition

A $[m_1, m_2, \ldots m_t]$-**factor** *of digraph* G *is a spanning subdigraph of* G *that is the disjoint union of* $\vec{C}_{m_1}, \vec{C}_{m_2}, \ldots, \vec{C}_{m_t}$.

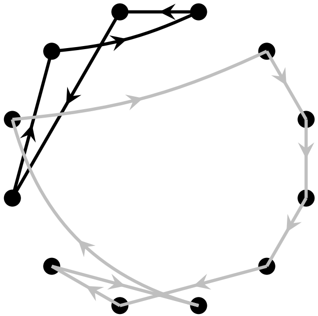

Figure: A $[4, 8]$-factor of K^*_{12}.

Definitions

Definition

A $[m_1, m_2, \ldots, m_t]$-**factorization** *of digraph* G *is a decomposition of* G *into* $[m_1, m_2, \ldots, m_t]$*-factors.*

The graph-theoretic formulation of the directed OP

Problem ($\mathrm{OP}^*(m_1, m_2, \ldots, m_t)$)

Let $m_1, m_2, \ldots, m_t \geqslant 2$. If $m_1 + m_2 + \ldots + m_t = n$, does K_n^* admit a $[m_1, m_2, \ldots, m_t]$-factorization?

If $m_1 = m_2 = \ldots = m_t = m$, then we write $\mathrm{OP}^*(m^t)$.

Background

Theorem (Bermond, Germa, and Sotteau (1979); Tillson (1980), Bennett and Zhang (1990); Adams and Bryant (Unpublished); Abel, Bennett, and Ge (2002); Burgess and Šajna (2014); Burgess, Francetić, and Šajna (2018); L-M (2024))

The $OP^*(m^t)$ has a solution except when

$$(m, t) \notin \{(3, 2), (4, 1), (6, 1)\}.$$

The directed OP has been completely resolved when all tables are of the same length.

Background

Theorem (Kadri and Šajna (2023+))

Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution except possibly when $m_1 \in \{4, 6\}$ and m_2 is even.

Idea: Take a solution to $\mathrm{OP}^*(m_1^1)$ and construct a solution to $\mathrm{OP}^*(m_1, m_2)$.

Problem: $\mathrm{OP}^*(4^1)$ and $\mathrm{OP}^*(6^1)$ do not have a solution.

Result

Theorem (Horsley and L-M (2023+))

Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution when $m_1 \in \{4, 6\}$ and m_2 is even.

We construct an $[m_1, m_2]$-factorization of K_n^* when $m_1 + m_2 = n$, $m_1 \in \{4, 6\}$, and m_2 is even.

Strategy when $n \equiv 2 \pmod 4$

Step 1: Decompose K_n^* into $\frac{n-3}{2}$ spanning subdigraphs that fall into one of two isomorphisms classes G_1 and G_2.

Step 2: Show that G_1 and G_2 both admit a $[m_1, m_2]$-factorization.

First class of digraphs

Objective: To construct a $[4, 10]$-factorization of K_{14}^*.

First class of digraphs

Objective: To construct a $[4, 10]$-factorization of K_{14}^*.

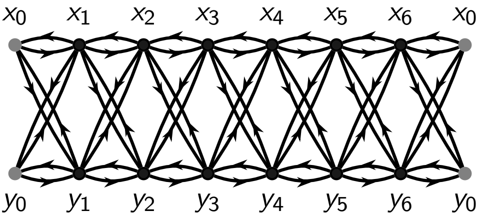

Figure: The first directed graph $G_1 = \vec{C}_7[2]$.

First class of digraphs

Objective: To construct a $[4, 10]$-factorization of K_{14}^*.

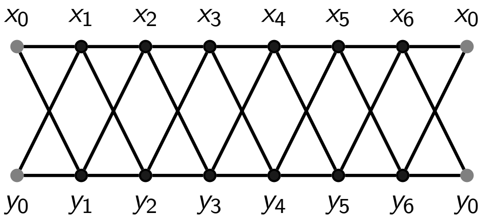

Figure: The underlying graph of $\vec{C}_7[2]$ written $C_7[2]$.

Easy result

Lemma (Häggkvist Lemma (Häggkvist (1985)))

Let $m_1, m_2, \ldots, m_t$ be even integers greater than 2. The graph $C_r[2]$ admits a undirected $[m_1, m_2, \ldots, m_t]$-factorization.

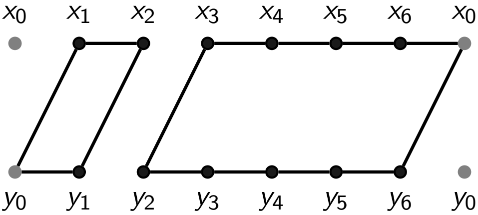

Figure: A undirected $[4, 10]$-factor of $C_7[2]$.

Easy result

Corollary

Let $m_1, m_2, \ldots, m_t$ be even integers greater than 2. The graph $\vec{C}_r[2]$ admits an $[m_1, m_2, \ldots, m_t]$-factorization.

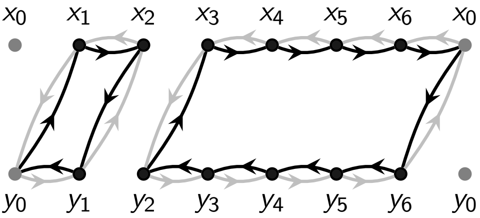

Figure: Two directed $[4, 10]$-factors of $\vec{C}_7[2]$.

Second spanning subdigraph

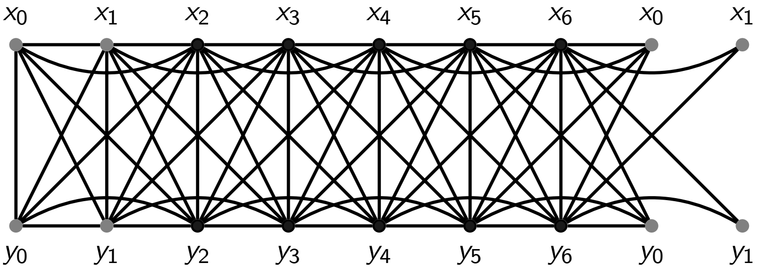

Figure: The underlying graph of G_2.

Each edge represents a pair of arcs, one for each direction.

Constructing a [4, 10]-factor.

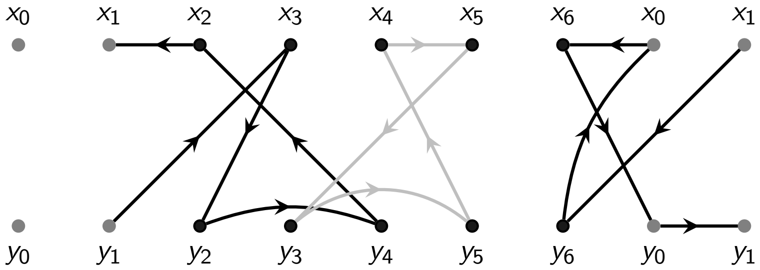

Figure: A [4, 10]-factor of G_2.

Extension: a simple guide

Step 1:

Extension: a simple guide

Step 2:

Extension: a simple guide

Step 3:

Extension

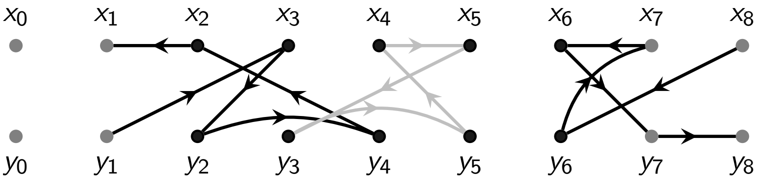

Figure: A $[4, 10]$-factor of G_2.

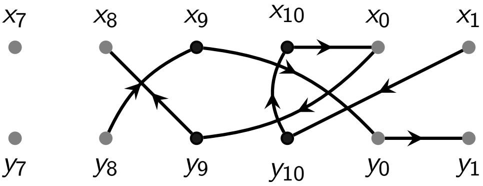

Figure: An extension of length 8.

Extension

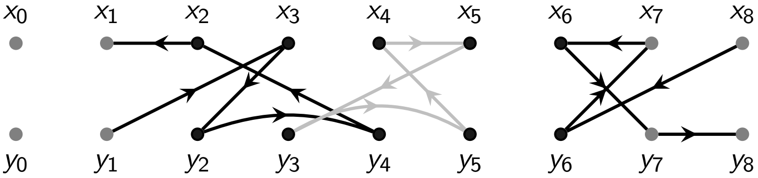

Figure: A $[4, 10]$-factor of G_2.

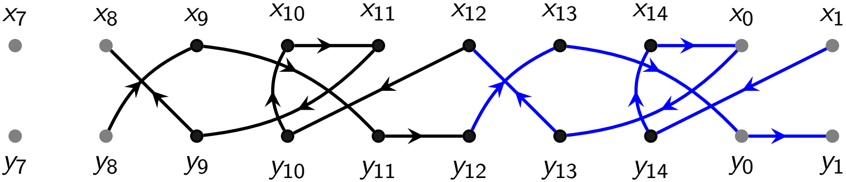

Figure: An extension of length 16.

Proposition

The digraph G_2 admits a $[m_1, m_2]$-factorization for $m_1 \in \{4, 6\}$ and $m_1 + m_2 \equiv 2 \pmod 4$.

The case $n \equiv 0 \pmod 4$

We obtain a decomposition of K_n^* into the following two digraphs:

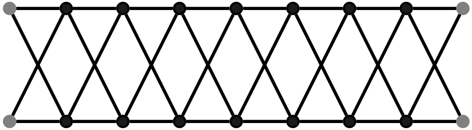

Figure: The underlying graph of G_1.

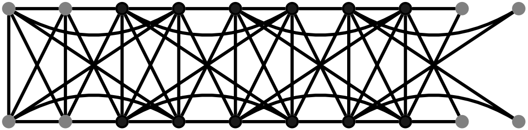

Figure: The underlying graph of G_2.

A complete solution

Theorem (Kadri and Šajna (2023+) and Horsley and L-M (2023+))

Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution.

Next step: To generalize our methods to obtain a solution to $\mathrm{OP}^*(m_1, m_2, \ldots, m_t)$ for any combinations of even $m_1, m_2, \ldots, m_t$.